

# Symmetries of BFKL Equation

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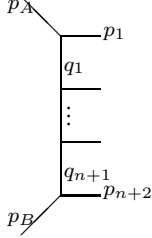
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## Abstract

We discuss the algebraic structure of the spin chains related to high energy scattering in QCD. We study the  $sl(2)$  Yangian symmetry and possible generalizations to nonzero spin and anisotropy parameter.

# 1 Introduction.

Since seventies, much evidence was obtained that the high energy scattering of hadrons can be described by exactly solvable model. It describes interaction of Reggeons – the collective coordinates, which appear in QCD in the Regge limit. The corresponding Hamiltonian was discovered in the paper [1]. The authors studied the high energy scattering of two vector particles in the theory with spontaneously broken gauge symmetry.



By means of unitarity, the scattering amplitude can be related to the integral of the product of two  $2 \rightarrow 2 + n$  amplitudes over the momenta of intermediate  $2 + n$  states, and sum over  $n$ . The regge limit is when  $s = (p_A + p_B)^2 \gg m^2$  and  $-(p_A - p_B)^2 \sim m^2$ . In this limit the main contribution from the integral over the intermediate states comes from the so called multiregge region, where

$$1 \gg -(p_B, q_1)/s \gg -(p_B, q_2)/s \gg \dots \gg -(p_B, q_{n+1})/s \sim m^2/s, \\ m^2/s \sim (p_A, q_1)/s \ll \dots \ll (p_A, q_n)/s \ll (p_A, q_{n+1})/s \ll 1 \quad (1)$$

One of the key observations was the specific form of the  $2 \rightarrow 2 + n$  amplitude in this limit:

$$A_{2 \rightarrow 2+n} = s \frac{(s_1/m^2)^{\alpha(q_1^2)}}{(q_1^2 - m^2)} \dots \frac{(s_{n+1}/m^2)^{\alpha(q_{n+1}^2)}}{(q_{n+1}^2 - m^2)} \\ \Gamma_{AD_0}^{i_1} \gamma_{i_1 i_2}^{D_1} \dots \gamma_{i_n i_{n+1}}^{D_n} \Gamma_{BD_{n+1}}^{i_{n+1}} \quad (2)$$

where

$$\alpha(k) = -\frac{g^2}{(2\pi)^3} \int d^2 q \frac{|k|^2 + m^2}{(|q-k|^2 + m^2)(|q|^2 + m^2)} = -\frac{g^2}{8\pi^2} \log \left( \frac{|k|^2}{m^2} \right) + o(m^2) \quad (3)$$

is related to the Regge trajectory  $j = 1 + \alpha(t)$  and

$$\gamma_{ij} = m\delta_{ij}, \quad \gamma_{ij}^D(q_1, q_2) = ig\epsilon_{ijD} \mathcal{P}_\mu(q_1, q_2) e_D^\mu \quad (4)$$

are the effective vertices [1] for emission of scalar and vector particles. If the particle  $D$  is on shell, *i.e.*  $(q_1 - q_2)^2 = m^2$ , the effective vertex equals:

$$\mathcal{P}^\mu(q_1, q_2) = -(q_1 + q_2)_\perp^\mu - \alpha_1 p_A^\mu \left( 1 - 2 \frac{|q_{1\perp}|^2 + m^2}{|q_{1\perp} - q_{2\perp}|^2 + m^2} \right) \\ + \beta_2 p_B^\mu \left( 1 - 2 \frac{|q_{2\perp}|^2 + m^2}{|q_{1\perp} - q_{2\perp}|^2 + m^2} \right) \quad (5)$$

The important property of the  $\mathcal{P}_\mu$  is that if the emitted particle is on shell, the scalar product of the vertices:

$$\sum_D \gamma_{ij}^D(k_1, p_1) \gamma_{i'j'}^D(k_2, p_2) = 2g^2 (K^{(0)} c_{ij} c_{i'j'} + K^{(1)} c_{ij}^k c_{i'j'}^k + K^{(2)} c_{ij}^{kl} c_{i'j'}^{kl}) \quad (6)$$

depends only on the components of momenta, orthogonal to the plane  $(p_A, p_B)$ . Here the coefficients  $c$  are the projectors on the space with definite isospin. In particular, the component of isospin 0 is:

$$K^{(0)} = -2|k_1 + k_2|^2 - 3m^2 + 2 \frac{(|k_1|^2 + m^2)(|p_2|^2 + m^2) + (|p_1|^2 + m^2)(|k_2|^2 + m^2)}{|k_1 - p_1|^2 + m^2} \quad (7)$$

This expression in the limit  $m \rightarrow 0$  goes to:

$$K^{(0)} \stackrel{m \rightarrow 0}{=} \frac{2}{|q|^2} (k_1 k_2^* p_1^* p_2 + \text{c.c.}) \quad (8)$$

Thus, the sum of the ladder diagrams <sup>1</sup>:

$$\begin{aligned} & \overline{p_0} \quad \overline{q_0} \quad + \quad \overline{p_0} \quad \overline{p_1} \quad \overline{k_1} \quad \overline{q_1} \quad + \quad \overline{\quad} \quad \overline{\quad} \quad \overline{\quad} \quad + \quad \dots = \\ & = \phi_s(q_0, p_0) = \sum_{n=0}^{\infty} \left( \frac{N_c g^2}{(2\pi)^3} \right)^n \frac{(s_0/m^2)^{\alpha(q_0)+\alpha(p_0)}}{(|q_0|^2+m^2)(|p_0|^2+m^2)} \times \\ & \times \prod_{j=1}^n \left[ \int d^2 k_{j\perp} \int \frac{d\alpha_j}{\alpha_j} \frac{2[q_j p_j^* q_{j-1}^* p_{j-1} + \text{h.c.}]}{|k_j|^2+m^2} \frac{(s_j/m^2)^{\alpha(q_j)+\alpha(p_j)}}{(|q_j|^2+m^2)(|p_j|^2+m^2)} \right] \end{aligned} \quad (9)$$

is described by the BFKL integral equation. Here  $q_j = q_{j-1} - k_j$ ,  $p_j = p_{j-1} + k_j$ . The mass  $m$  plays the role of regulator.  $N_c$  is the number of colors. We denoted  $q_j$ ,  $p_j$  the  $\perp$ -components of the momenta  $q_j^\mu$  and  $p_j^\mu$ , and the  $\parallel$ -components of  $q_j^\mu$  are expressed through the Sudakov's variables:

$$q_{j\parallel} = \alpha_j p_A - \beta_j p_B$$

$|q|^2$  means  $-q_\perp^2$ . The integration is over the momenta of the intermediate particles, which are on shell. Notice that (in the Regge limit):

$$\begin{aligned} s_j &= (q_{j-1}^\mu - q_{j+1}^\mu)^2 = -(\alpha_{j-1} - \alpha_{j+1})(\beta_{j-1} - \beta_{j+1})s - \\ &- |q_{j-1} - q_{j+1}|^2 = \alpha_{j-1}\beta_{j+1}s - |q_{j-1} - q_{j+1}|^2 \end{aligned} \quad (10)$$

On the other hand, from the mass-shell condition for the emitted particle follows  $\beta_j s = \frac{1}{\alpha_{j-1}}(m^2 + |q_{j-1} - q_j|^2)$ . Thus  $s_j = \frac{\alpha_{j-1}}{\alpha_j}(m^2 + |q_j - q_{j+1}|^2)$

In particular,

$$\prod_{j=0}^n \frac{s_j}{m^2} = \frac{1}{\alpha_n} \prod_{j=0}^n \left( 1 + \frac{|q_j - q_{j+1}|^2}{m^2} \right) = \frac{s}{m^2} \left( 1 + \frac{|k_1|^2}{m^2} \right) \left( 1 + \frac{|k_2|^2}{m^2} \right) \dots \left( 1 + \frac{|k_n|^2}{m^2} \right) \quad (11)$$

In the LLA, the product  $(1 + |k_n|^2/m^2)$  may be omitted. The formula  $\frac{s}{m^2} = \prod \frac{s_i}{m^2}$  enables to interpret  $\log \frac{s}{m^2}$  as time in the corresponding integrable system:  $\tau_j = \log(s_j/m^2)$ . We can also write the integration measure  $\prod_j \frac{d\alpha_j}{\alpha_j}$  as  $\prod_j d\tau_j$ . Thus, the equation (9) can be written as:

$$\begin{aligned} \phi_\tau(q, p) &= \sum_{n=0}^{\infty} \left[ \frac{N_c g^2}{(2\pi)^3} \right]^n 2^n \int_{\tau > \tau_n > \dots > \tau_1 > 0} d\tau_n \dots d\tau_1 e^{[\alpha(q)+\alpha(p)](\tau-\tau_n)} \times \\ &\times \prod_{j=1}^n \left\{ \left[ \frac{1}{q^* p} \int \frac{d^2 k_j}{|k_j|^2+m^2} e^{k_j(\partial_p - \partial_q) + k_j^*(\partial_p^* - \partial_q^*)} q^* p + \text{h.c.} \right] e^{[\alpha(q)+\alpha(p)](\tau_j - \tau_{j-1})} \right\} \frac{1}{|q|^2 |p|^2} \end{aligned} \quad (12)$$

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<sup>1</sup>In the LLA, the main contribution to the integral over the intermediate states comes from the multiregion region.

If we denote  $ix = \partial_p$ ,  $iy = \partial_q$ , and take the integrals over  $dk_j$ , we get:

$$\phi_\tau(q, p) = \exp\left(-\tau \frac{N_c g^2}{8\pi^2} \hat{H}\right) \frac{N_c g^2 / (2\pi)^3}{|q|^2 |p|^2} \quad (13)$$

where<sup>2</sup>

$$\hat{H} = \log(|q|^2) + \log(|p|^2) + \frac{1}{qp^*} \log(|x - y|^2) qp^* + \frac{1}{q^*p} \log(|x - y|^2) q^*p - 4\psi(1) \quad (14)$$

This hamiltonian is the sum of two expressions, acting on holomorphic and antiholomorphic functions.

The part of the scattering amplitude, corresponding to the exchange of pomeron, will contain the factor  $\exp\left[-h \frac{N_c g^2}{8\pi^2} \log s\right]$  where  $h$  is the corresponding eigenvalue of  $\hat{H}$ .

In this paper we discuss the algebraic properties of the Hamiltonian (14). We review the derivation of this Hamiltonian from the transfer-matrix. We will show, how it commutes with the  $sl(2)$  Yangian, and discuss its various forms. In the last section we will consider the integrable perturbation of this Hamiltonian, which on the language of spin chains corresponds to turning on small anisotropy.

## 2 Spin chains.

We want to construct an integrable spin chain such that the spin- $j$  representation of  $sl(2)$  is associated with each site. This system will have an infinite number of commuting hamiltonians. In the framework of Quantum Inverse Scattering Method, these hamiltonians are included into the noncommutative algebra. This algebra is generated by the entries of the monodromy matrix  $T(\lambda)$ , which acts in the auxiliary vector space. The commutation relations are encoded in the  $RTT$  equations:

$$R(\lambda - \mu)T(\lambda)T(\mu) = T(\mu)T(\lambda)R(\lambda - \mu) \quad (15)$$

Where  $R$  is the (numerical)  $R$ -matrix. An important property of such algebras is the existence of "comultiplication". Namely, if we have two matrices  $T_1(\lambda)$ ,  $T_2(\lambda)$ , satisfying the  $RTT$  relation, then their product  $T_1(\lambda) \otimes T_2(\lambda)$  also satisfies these relations. Schematically, if

$$\begin{array}{c} \text{then} \\ \begin{array}{ccc} \begin{array}{c} \text{Diagram 1: A vertical line on the left, a horizontal line crossing it, and a vertical line on the right. The horizontal line has a loop on the left side.} \\ \text{Diagram 2: A horizontal line on the left, a vertical line crossing it, and a horizontal line on the right. The vertical line has a loop on the top side.} \end{array} = \begin{array}{c} \text{Diagram 3: A horizontal line on the left, a vertical line crossing it, and a horizontal line on the right. The vertical line has a loop on the bottom side.} \\ \text{Diagram 4: A vertical line on the left, a horizontal line crossing it, and a vertical line on the right. The horizontal line has a loop on the right side.} \end{array} \end{array}$$

This suggests the way to construct the monodromy matrix and the hamiltonians: one finds some simplest representation of  $RTT$ -algebra (which corresponds to one site of the chain), and then constructs the  $T$ -operator for the whole chain using the comultiplication.

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<sup>2</sup>Here we followed the paper [1]. See [3] for another derivation, using the lagrangian language.

For example, the  $RTT$  relations are satisfied by the entries of the  $2 \times 2$  matrix

$$L_{elem.}(\lambda) = 1 + \sum_{a=1}^3 \frac{1}{\lambda} \sigma_a S^a \quad (16)$$

where  $S^a$  are the generators of  $sl(2)$  in the spin- $j$  representation. Taking the product of these "elementary"  $L$ -operators over all the sites, we get some  $2 \times 2$  matrix, whose entries are the operators acting in the Hilbert space of the system. The traces of this matrix will commute for different values of the spectral parameter. Unfortunately, the trace of  $L$ , considered as the power series in the spectral parameter, will have nonlocal operators as coefficients, thus we cannot consider them as hamiltonians.

The way out is to consider more complicated object, "the fundamental  $L$ -operator", associated with each site. The fundamental  $L$ -operator is the matrix in  $2j+1$ -dimensional "auxiliary" space, whose entries are the operators acting in the local space on the site. We need something like

$$L = \lambda + \sum_{a=-j}^j S_{aux}^a S_q^a \quad (wrong)$$

but such a simple  $L$  would not satisfy any  $RTT$  relation, and more care is needed to construct them correctly. The fundamental  $L$ -operators formally coincide with the fundamental  $R$ -matrix intertwining them.

The  $RTT$  relations ensure the integrability of the system [6, 7, 8, 10].

In the next section we review the construction of the fundamental  $R$ -matrix and derive the hamiltonian of the solvable spin chain. Actually we will consider the case of XXZ model, where the symmetry is not  $sl(2)$ , but  $U_q(sl(2))$ , the quantum algebra. The hamiltonian depends on  $q$  – the deformation parameter.

### 3 Fusion procedure. The fundamental R-matrix.

The idea [11, 12, 14] is to consider first the  $R$ -matrix for the spin-1/2 chain, study the symmetry properties of this  $R$ -matrix and then to construct the spin- $j$   $R$ -matrix from it. Remember that the spin-1/2 matrix is related to the 6-vertex model statistical weights,

$$\begin{bmatrix} \sin(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \sin(\lambda) & \sin(\eta) & 0 \\ 0 & \sin(\eta) & \sin(\lambda) & 0 \\ 0 & 0 & 0 & \sin(\lambda + \eta) \end{bmatrix} \quad (17)$$

where  $\eta$  is anisotropy parameter. Introduce  $z = e^{i\lambda}$ ,  $q = e^{i\eta}$ , and consider the modified  $2 \times 2$   $R$ -matrix, which differs from the original one by conjugation:

$$\begin{aligned} R'_{4 \times 4}(z) &= z^{\frac{1}{2}\sigma^3 \otimes 1} R(z) z^{-\frac{1}{2}\sigma^3 \otimes 1} = \\ &= \begin{bmatrix} zq - \frac{1}{zq} & 0 & 0 & 0 \\ 0 & z - \frac{1}{z} & z(q - \frac{1}{q}) & 0 \\ 0 & \frac{1}{z}(q - \frac{1}{q}) & z - \frac{1}{z} & 0 \\ 0 & 0 & 0 & zq - \frac{1}{zq} \end{bmatrix} \end{aligned} \quad (18)$$

The purpose of this conjugation is to ensure that the  $4 \times 4$   $R$ -matrix satisfies the following symmetry properties:

$$\begin{aligned} & (q^{\frac{\sigma^3}{2}} \otimes \sigma^\pm + \sigma^\pm \otimes q^{-\frac{\sigma^3}{2}}) R'(z) = \\ & = R'(z) (q^{-\frac{\sigma^3}{2}} \otimes \sigma^\pm + \sigma^\pm \otimes q^{\frac{\sigma^3}{2}}) \end{aligned} \quad (19)$$

and

$$\begin{aligned} & \left( q^{-\frac{\sigma^3}{2}} \otimes \frac{\sigma^+}{z} + \frac{\sigma^+}{w} \otimes q^{\frac{\sigma^3}{2}} \right) R' \left( \sqrt{\frac{z}{w}} \right) = \\ & = R' \left( \sqrt{\frac{z}{w}} \right) \left( \frac{\sigma^+}{w} \otimes q^{-\frac{\sigma^3}{2}} + q^{\frac{\sigma^3}{2}} \otimes \frac{\sigma^+}{z} \right) \end{aligned} \quad (20)$$

The first of these properties means, that the  $R$ -matrix is invariant under the quantum group  $\mathcal{U}_q(sl(2))$ . The second extends this symmetry to  $\mathcal{U}_q(\widehat{sl}(2))$ , – the quantum Kac-Moody algebra. (See Appendix.)

Let us show that there are  $R$ -matrices acting in the tensor product of spin- $j$  ( $j > 1/2$ ) representations, which satisfy the similar symmetry properties. Indeed, we may construe the  $R$ -matrix for, say, spin -1, by means of the following "fusion procedure":

$$R' \left( \frac{z}{w} \right) = \begin{array}{cc|c} R'_{4 \times 4} \left( \frac{z}{qw} \right) & R'_{4 \times 4} \left( \frac{z}{w} \right) & 2 \\ \hline R'_{4 \times 4} \left( \frac{z}{w} \right) & R'_{4 \times 4} \left( \frac{qz}{w} \right) & 1 \\ \hline -2 & -1 & \end{array}$$

(The matrices are multiplied from the bottom right to the top left.) Notice the special relations between the arguments: they differ by multiplication on the power of  $q$ . The reason is that we want the tensor products of  $R$ 's to act correctly on the symmetric part of the tensor product of two-dimensional spaces. For example, consider the product  $R' \left( \frac{z}{w} \right) \otimes R' \left( \frac{qz}{w} \right)$  (the two rightmost  $R$ 's on the figure). They act on the tensor product  $1 \otimes 2$ . Notice that the "q-symmetric" tensors<sup>3</sup> remain q-symmetric. Indeed, the antisymmetrization  $A_{12}$  of  $1 \otimes 2$  can be expressed through the  $R$ -matrix:  $A_{12}^{(q)} = q^{-\frac{1}{2}\sigma^3 \otimes 1} A_{12} q^{\frac{1}{2}\sigma^3 \otimes 1} = \frac{1}{q^{-1}-q} R'(q^{-1})$ . Thus, using the Yang-Baxter relations  $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ , we have:

$$\begin{aligned} & A_{12}^{(q)} R'_{2,-1} \left( \frac{z}{w} \right) R'_{1,-1} \left( q \frac{z}{w} \right) = \\ & = R'_{1,-1} \left( q \frac{z}{w} \right) R'_{2,-1} \left( \frac{z}{w} \right) A_{12}^{(q)} \end{aligned} \quad (21)$$

so the space of q-symmetric tensors is preserved. The matrix  $R' \left( \frac{z}{w} \right)$  acts in the tensor product

$$\text{Sym}_q(\mathbf{C}^2)^{\otimes n} \otimes \text{Sym}_q(\mathbf{C}^2)^{\otimes n}$$

The space  $\text{Sym}_q(\mathbf{C}^2)^{\otimes n}$  is an irreducible representation of quantum  $sl(2)$ , and it also can be considered as the irreducible part of

$$\rho_{z^2}^{1/2} \otimes \rho_{q^2 z^2}^{1/2} \otimes \cdots \otimes \rho_{q^{2(n-1)} z^2}^{1/2}$$

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<sup>3</sup> q-symmetric are the tensors which become symmetric after action of  $q^{\frac{1}{2}\sigma^3 \otimes 1}$

as described in Appendix. This yields the desired generalization of the symmetry properties of fundamental R-matrix:

$$\left[ PR' \left( \sqrt{\frac{z}{w}} \right), \left( q^{-S_3} \otimes E + E \otimes q^{S_3} \right) \right] = 0 \quad (22)$$

and

$$\begin{aligned} PR' \left( \sqrt{\frac{z}{w}} \right) \left( q^{S_3} \otimes \frac{E}{z} + \frac{E}{w} \otimes q^{-S_3} \right) &= \\ &= \left( q^{S_3} \otimes \frac{E}{w} + \frac{E}{z} \otimes q^{-S_3} \right) PR' \left( \sqrt{\frac{z}{w}} \right) \end{aligned} \quad (23)$$

Notice that the Yang-Baxter equations for  $R'$  can be derived from the explicit construction from the Yang-Baxter equations for  $4 \times 4$  R-matrix, as well as from the consistency of these symmetry properties.

From (22) we infer that:

$$PR' \left( \sqrt{\frac{z}{w}} \right) = \sum_{p=0}^n F_p(z/w) P_p \quad (24)$$

where  $P_p$  is the projector on the subspace in the tensor product, which has spin  $p$ .

The coefficients  $F_p$  may be found from (23).

For example, one may consider the action of (23) on the highest weight vector:

$$\begin{aligned} \left| \left( \frac{n}{2}, \frac{n}{2} \right)_p, p \right\rangle &= \frac{1}{\sqrt{(n-p)_q(p+1)_q}} \left| \frac{n}{2}, \frac{n}{2} \right\rangle \otimes \left| \frac{n}{2}, p - \frac{n}{2} \right\rangle - \\ &- \frac{q^{-p-1}}{\sqrt{(n)_q}} \left| \frac{n}{2}, \frac{n}{2} - 1 \right\rangle \otimes \left| \frac{n}{2}, p - \frac{n}{2} + 1 \right\rangle + \dots \end{aligned} \quad (25)$$

The key point is that the operator  $\left( q^{S_3} \otimes \frac{E}{z} + \frac{E}{w} \otimes q^{-S_3} \right)$  is the tensor one – it follows from the Serre relations, see Appendix. Thus, when we act by this operator on  $\left| \left( \frac{n}{2}, \frac{n}{2} \right)_p, p \right\rangle$ , we get spins not higher than  $p+1$ . The coefficient of  $\left| \left( \frac{n}{2}, \frac{n}{2} \right)_{p+1}, p+1 \right\rangle$  can be found from

$$\begin{aligned} &\left( q^{S_3} \otimes \frac{E}{z} + \frac{E}{w} \otimes q^{-S_3} \right) \left| \left( \frac{n}{2}, \frac{n}{2} \right)_p, p \right\rangle = \\ &= \left( \frac{q^{n/2}}{z} - \frac{q^{-2p-2+\frac{n}{2}}}{w} \right) \left| \frac{n}{2}, \frac{n}{2} \right\rangle \otimes \left| \frac{n}{2}, p - \frac{n}{2} + 1 \right\rangle + \dots \end{aligned} \quad (26)$$

We get:

$$F_{p+1}(z/w) \left( \frac{q^{n/2}}{z} - \frac{q^{-2p-2+\frac{n}{2}}}{w} \right) = F_p(z/w) \left( \frac{q^{n/2}}{w} - \frac{q^{-2p-2+\frac{n}{2}}}{z} \right) \quad (27)$$

– the relation between  $F_p$  and  $F_{p+1}$ . Finally, we get the expression for  $R'$ :

$$PR'(z) = P_0 - P_1 q^{-2} \frac{1-z^2 q^2}{1-z^2 q^{-2}} + P_2 q^{-6} \frac{1-z^2 q^2}{1-z^2 q^{-2}} \frac{1-z^2 q^4}{1-z^2 q^{-4}} - \dots \quad (28)$$

This expression can be rewritten in terms of Gamma-functions, which enables to generalize it for general (non-integer)  $j$ <sup>4</sup>:

$$PR'(e^{i\lambda}) = (-1)^J q^{-J(J+1)} \frac{\Gamma_{q^2}(\frac{\lambda}{\eta} + J + 1) \Gamma_{q^2}(\frac{\lambda}{\eta} - J)}{\Gamma_{q^2}(\frac{\lambda}{\eta} + 1) \Gamma_{q^2}(\frac{\lambda}{\eta})} \quad (29)$$

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<sup>4</sup>We consider the product of two spin- $j$  representations. The capital  $J$  denotes the operator  $\sum p F_p$ .

$PR'(1) = 1$ . The rational limit corresponds to taking  $\lambda$  and  $\eta$  very small and keeping their ratio finite.

The hamiltonian of the integrable model is related to the R-matrix:

$$R(z = 1 + \lambda) = P + \lambda PH + o(\lambda) \quad (30)$$

where the symmetric  $R$ -matrix is related to the  $R$ -matrix which we constructed in fusion procedure by conjugation:

$$R(z) = gR'(z)g^{-1}, \quad g = \prod_{k<0} z^{\frac{-\sigma_k^3}{2}} \prod_{k<0} q^{-(n+k+1)\frac{\sigma_k^3}{2}} \prod_{k>0} q^{-k\frac{\sigma_k^3}{2}} \quad (31)$$

– the numeration of the vector spaces as on the figure. Notice that the last two products in  $g$ ,  $\prod_{k<0} q^{-(n+k+1)\frac{\sigma_k^3}{2}} \prod_{k>0} q^{-k\frac{\sigma_k^3}{2}}$  are symmetric (commute with  $P$ ). They ensure that  $R$  acts on symmetric tensors (not q-symmetric). The first product,  $g_1 = \prod_{k<0} z^{\frac{-\sigma_k^3}{2}}$ , ensures that  $R(z)$  is symmetric. One can see it from the fusion procedure, or from the symmetry properties, eqs. (22), (23). Indeed, conjugating these equations by  $g_1$ , we get:

$$\begin{aligned} PR\left(\sqrt{\frac{z}{w}}\right) \left(q^{-S_3} \otimes \sqrt{z}E + \sqrt{w}E \otimes q^{S_3}\right) &= \\ = \left(q^{-S_3} \otimes \sqrt{w}E + \sqrt{z}E \otimes q^{S_3}\right) PR\left(\sqrt{\frac{z}{w}}\right) \end{aligned} \quad (32)$$

and

$$\begin{aligned} PR\left(\sqrt{\frac{z}{w}}\right) \left(q^{S_3} \otimes \sqrt{w}E + \sqrt{z}E \otimes q^{-S_3}\right) &= \\ = \left(q^{S_3} \otimes \sqrt{z}E + \sqrt{w}E \otimes q^{-S_3}\right) PR\left(\sqrt{\frac{z}{w}}\right) \end{aligned} \quad (33)$$

– these equations are related to each other by permutation of tensor multipliers. If  $R$  were not symmetric, we could construct the symmetric solution as  $R + PRP$ . But these equations determine  $R$  uniquely. Thus,  $R$  is symmetric.

From (30) we get the hamiltonian:

$$\begin{aligned} P + \lambda PH + o(\lambda) &= P + \lambda \left( \sum_{k>0} \frac{\sigma_k^3}{2} - \sum_{k<0} \frac{\sigma_k^3}{2} \right) P + \\ &+ \frac{\lambda}{\eta} P(\psi_{q^2}(J+1) + \psi_{q^2}(-J)) + o(\lambda) \end{aligned} \quad (34)$$

We know from construction, that this expression is symmetric. The hamiltonian is:

$$\begin{aligned} H &= \eta \left( \sum_{k<0} \frac{\sigma_k^3}{2} - \sum_{k>0} \frac{\sigma_k^3}{2} \right) + (\psi_{q^2}(J+1) + \psi_{q^2}(-J)) = \\ &= \frac{1}{2} (\psi_{q^2}(J+1) + \psi_{q^2}(-J) + P\psi_{q^2}(J+1)P + P\psi_{q^2}(-J)P) \end{aligned} \quad (35)$$

And following the tradition in quantum group theory, we denote

$$\hbar = i\eta \quad (36)$$



## 4 The rational case: Yangian symmetry.

Here we consider separately the case  $\hbar = 0$ . In this case instead of the quantum Kac-Moody symmetry, we have the Yangian symmetry [13]. It can be derived from the basic property of the fundamental  $R$ -matrix: it intertwines  $L$ -operators. Namely, the entries of the  $2 \times 2$  matrix

$$L(\lambda) = 1 + \frac{1}{\lambda} \vec{\sigma} \cdot \vec{S} \quad (37)$$

satisfy the commutation relations of  $RTT$  algebra:

$$L_1(\lambda)L_2(\lambda + \mu)R_{12}(\mu) = R_{12}(\mu)L_2(\lambda + \mu)L_1(\lambda) \quad (38)$$

Consider this equation as a series in  $\lambda^{-1}$ . The zeroth order term is trivial, the first order term expresses the fact that  $R$  is  $sl(2)$ -invariant. And the coefficient of  $\frac{1}{\lambda^2}$  is:

$$\begin{aligned} & (\vec{\sigma} \cdot \vec{S}_1)(\vec{\sigma} \cdot \vec{S}_2)R_{12}(\mu) - \mu(\vec{\sigma} \cdot \vec{S}_2)R_{12}(\mu) = \\ & = R_{12}(\mu)(\vec{\sigma} \cdot \vec{S}_2)(\vec{\sigma} \cdot \vec{S}_1) - \mu R_{12}(\mu)(\vec{\sigma} \cdot \vec{S}_2) \end{aligned} \quad (39)$$

Consider this equation up to first order in  $\mu$ ,  $R_{12}(\mu) = P_{12} + \mu P_{12}H_{12} + \dots$ :

$$i[[\vec{S}_1 \times \vec{S}_2], H_{12}] = \vec{S}_2 - \vec{S}_1 \quad (40)$$

Here  $H_{12}$  is the two-site hamiltonian. If we had a chain with  $n$  sites, with the hamiltonian

$$\sum_{j=1}^n H_{j,j+1} \quad (41)$$

then we would get:

$$[\sum_{i<j} \vec{S}_i \times \vec{S}_j, \sum_{j=1}^n H_{j,j+1}] = \vec{S}_n - \vec{S}_1 \quad (42)$$

The operators  $\vec{T} = \sum_{i<j} \vec{S}_i \times \vec{S}_j$  commute with the hamiltonian modulo the boundary terms<sup>5</sup>. Together with the generators of  $sl(2)$  they generate the algebra called Yangian.

We want to construct an integrable spin chain with arbitrary spin in the sites. We will use the functional representation of  $sl(2)$ :

$$\begin{aligned} S^+ &= z^2 \partial_z - 2jz, \\ S^3 &= z \partial_z - j, \\ S^- &= -\partial_z \end{aligned} \quad (43)$$

In this representation:

$$\mathcal{T}^- = \sum_{i<j} [(z_i - z_j) \partial_i \partial_j + j(\partial_i - \partial_j)] \quad (44)$$

and the other  $\mathcal{T}$ 's may be obtained from  $\mathcal{T}^-$  by commutation with  $sl(2)$ .

Let us construct the Hamiltonian which respects these symmetries. It is useful to go to the Fourier transform

$$f(z) = \int e^{ikz} f(k) dk \quad (45)$$

---

<sup>5</sup>One cannot define  $\vec{T}$  for the periodic chain, since the *linear* order is required, not just a cyclic one.

and write an ansatz for the hamiltonian as:

$$H\phi(p_1, p_2) = \int dk_1 dk_2 D(p_1, p_2; k_1, k_2) \phi(k_1, k_2) \quad (46)$$

Let us require that the commutator with  $\mathcal{T}_-$  is a local operator. Then in the bulk of the integration region:

$$\begin{aligned} & \left[ k_1 k_2 \left( \frac{\partial}{\partial k_1} - \frac{\partial}{\partial k_2} \right) + p_1 p_2 \left( \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right) + \right. \\ & \left. + (j+1)(p_1 - p_2) - j(k_1 - k_2) \right] D(p_1, p_2; k_1, k_2) = 0 \end{aligned} \quad (47)$$

The general solution of this equation is

$$D(p_1, p_2; k_1, k_2) = \frac{(k_1 k_2)^j}{(p_1 p_2)^{j+1}} D_0 \left( \frac{k_2 p_1}{k_1 p_2}; k_1 + k_2, p_1 + p_2 \right) \quad (48)$$

Now we use the  $sl(2)$ -symmetry. From the commutation with  $s^-$  we infer, that  $D_0$  should contain  $\delta(p_1 + p_2 - k_1 - k_2)$ . From the commutation with  $s^3$  we know how  $D_0$  scales with momenta, and find the dependence on  $k_1 + k_2$ :

$$D_0 = (k_1 + k_2) \delta(p_1 + p_2 - k_1 - k_2) \tilde{D}_0 \left( \frac{k_2 p_1}{k_1 p_2} \right)$$

To find  $\tilde{D}_0$ , we have to consider the commutation with  $s^+$ . We have:

$$\begin{aligned} H\phi(p_1, p_2) &= (p_1 + p_2) \int dk \frac{(p_1 - k)^j (p_2 + k)^j}{p_1^{j+1} p_2^{j+1}} \tilde{D}_0 \left( \frac{1+k/p_2}{1-k/p_1} \right) \phi(p_1 - k, p_2 + k) = \\ &= \int dk \frac{1}{k} (1 - k/p_1)^{j+1/2} (1 + k/p_2)^{j+1/2} F \left( \frac{1+k/p_2}{1-k/p_1} \right) \phi(p_1 - k, p_2 + k) \end{aligned} \quad (49)$$

where we have put  $F(z) = (z^{1/2} + z^{-1/2}) \tilde{D}_0$ . Acting on this integral by

$$s^+ = p_1 \frac{\partial^2}{\partial p_1^2} + p_2 \frac{\partial^2}{\partial p_2^2} + 2(j+1) \left( \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} \right) \quad (50)$$

and integrating by parts in  $\int dk$ , we get an equation for  $f$ :

$$\begin{aligned} & \left[ p_1 \frac{\partial^2}{\partial p_1^2} + p_2 \frac{\partial^2}{\partial p_2^2} - \frac{1}{k} \left( p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} + 1 \right) \left( p_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial p_2} \right) + \right. \\ & \left. + 2(j+1) \left( \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} \right) \right] (1 - k/p_1)^{j+1/2} (1 + k/p_2)^{j+1/2} F \left( \frac{1+k/p_2}{1-k/p_1} \right) = 0 \end{aligned} \quad (51)$$

or, introducing  $x = 1 - k/p_1$  and  $y = 1 + k/p_2$ :

$$\begin{aligned} & \left[ -x(1-x)^2 \frac{\partial^2}{\partial x^2} + y(1-y)^2 \frac{\partial^2}{\partial y^2} + \right. \\ & \left. + 2j \left( (1-x)^2 \frac{\partial}{\partial x} - (1-y)^2 \frac{\partial}{\partial y} \right) \right] (xy)^{j+1/2} F(x/y) = 0 \end{aligned} \quad (52)$$

or

$$\partial_x \partial_y [(xy)^{j+1/2} F(x/y)] = 0 \quad (53)$$

which means that either

$$(xy)^{j+1/2} F(x/y) = x^{2j+1}$$

or

$$(xy)^{j+1/2}F(x/y) = y^{2j+1}$$

Since the hamiltonian should be symmetric, we have to add these solutions:

$$H.\phi(p_1, p_2) = \int \frac{dk}{k} \left[ \left(1 - \frac{k}{p_1}\right)^{1+2j} + \left(1 + \frac{k}{p_2}\right)^{1+2j} \right] \phi(p_1 - k, p_2 + k) \quad (54)$$

Introducing  $ix_{12} = \partial/\partial p_1 - \partial/\partial p_2$ , we get:

$$H = \int \frac{dk}{k} \left[ \frac{1}{p_1^{1+2j}} e^{-ikx_{12}} p_1^{1+2j} + \frac{1}{p_2^{1+2j}} e^{-ikx_{12}} p_2^{1+2j} \right] \quad (55)$$

The integral over  $k$  is divergent and the cutoff is needed. From the  $s^3$  invariance we see, that the cutoff should be of the form  $\epsilon p_1$  or  $\epsilon p_2$ . The corresponding hamiltonian will be

$$H = \log(p_1 p_2) + \frac{1}{p_1^{1+2j}} \log(x_1 - x_2) p_1^{1+2j} + \frac{1}{p_2^{1+2j}} \log(x_1 - x_2) p_2^{1+2j} \quad (56)$$

Indeed,

$$\left[ z_1^2 \frac{\partial}{\partial z_1} + z_2^2 \frac{\partial}{\partial z_2} - 2j(z_1 + z_2), H \right] = 0 \quad (57)$$

and

$$[(z_1 - z_2)\partial_1\partial_2 - j(\partial_2 - \partial_1), H] = \partial_2 - \partial_1 \quad (58)$$

Notice that naively from the equation (47) it follows that the Hamiltonian exactly commutes with  $\mathcal{T}^-$ . The origin of the boundary terms in commutation relation with Yangian generator  $\mathcal{T}^-$  is an infrared divergency in the Fourier integral. That's why the RHS has such a simple form, local in the momentum space. It turns out that this divergency does not affect commutation with  $s^+$  [2].

Let us illustrate how to derive this hamiltonian (in a slightly different form) directly from (35). Let us first restrict ourselves with the case  $j = 0$ . From the representation for  $\psi$ -function:

$$\psi(z) = -\sum_{s=0}^{\infty} \frac{1}{s+z} + \text{const} \quad (59)$$

we get:

$$\begin{aligned} H &= \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)+(z_1-z_2)^2\partial_1\partial_2} = \\ &= \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)+x_1^2\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2}\right)} \end{aligned} \quad (60)$$

where  $x_0 = z_1 + z_2$  and  $x_1 = z_1 - z_2$ . This expression is equal to

$$H(\vec{n} \cdot \vec{n}') = 4\pi \sum_{l=0}^{\infty} x_1 \cdot \frac{\sum_{m=-l}^l Y_{lm}^*(\vec{n}') Y_{lm}(\vec{n})}{\frac{l(l+1)}{x_1^2} - \frac{\partial^2}{\partial x_1^2} - \frac{2}{x_1} \frac{\partial}{\partial x_1} + \frac{\partial^2}{\partial x_0^2}} \cdot \frac{1}{x_1^3} \quad (61)$$

when  $\vec{n} = \vec{n}'$ . The last expression is the Green function for the Laplace operator in 4 dimensions,

$$\begin{aligned} &rG(t, t'; r, r'; \vec{n}, \vec{n}')(r')^{-3} = \\ &= \frac{r}{(r')^3} \frac{\delta(|\vec{r}-\vec{r}'|-t+t') + \delta(|\vec{r}-\vec{r}'|+t-t')}{2|\vec{r}-\vec{r}'|} \end{aligned} \quad (62)$$

The expression  $\frac{1}{|x-y|}$ , properly regularized, is the kernel of the operator

$$-2 \log \partial - 2 \log z$$

This gives us the second form of the hamiltonian:

$$H = 2 \log(z_1 - z_2) + (z_1 - z_2) \log(\partial_1 \partial_2) (z_1 - z_2)^{-1} \quad (63)$$

From the explicit expression for  $J(J+1)$ :

$$J(J+1) = -(z_1 - z_2)^2 \partial_1 \partial_2 - 2j(z_1 - z_2)(\partial_1 - \partial_2) + 2j(2j+1) \quad (64)$$

we see, that  $J(J+1)$  for  $j \neq 0$  is conjugate to  $J(J+1)$  for  $j = 0$ , and consequently the hamiltonians for different spins are conjugate:

$$H_J = (z_1 - z_2)^{2j} H_0 (z_1 - z_2)^{-2j} \quad (65)$$

Finally, let us write down two forms of hamiltonian for general  $j$ :

$$\begin{aligned} H_j &= \log(\partial_1 \partial_2) + \frac{1}{\partial_1^{1+2j}} \log(z_1 - z_2) \partial_1^{1+2j} + \frac{1}{\partial_2^{1+2j}} \log(z_1 - z_2) \partial_2^{1+2j} = \\ &= 2 \log(z_1 - z_2) + (z_1 - z_2)^{1+2j} \log(\partial_1 \partial_2) (z_1 - z_2)^{-1-2j} \end{aligned} \quad (66)$$

The Hamiltonian for general  $J$  may correspond to taking into account some logarithmic corrections. The similar (but essentially different) Hamiltonian was considered in [9].

## 5 Trigonometric case for small $\hbar$ .

In this section we will calculate the hamiltonian in the anisotropic ( $q \neq 1$ ) case to first nontrivial order in  $\hbar$  – the deformation parameter. We will restrict ourselves with spin zero. Remember that the Hamiltonian is

$$\frac{1}{2} (\psi_{q^2}(-J) + \psi_{q^2}(J+1) + (\hbar \rightarrow -\hbar)) \quad (67)$$

The operator  $J$  is the quantum Casimir operator in the product of two representations. These two representations are spin-0 representations of quantum group in the space of functions, whose arguments we will denote  $z$  and  $w$ . The  $\mathcal{U}_q(sl(2))$  operators act as follows:

$$\begin{aligned} t &= q^{-2d} \\ e &= \frac{1}{x} \frac{q^d - q^{-d}}{q - q^{-1}} \\ f &= x \frac{q^{-d} - q^d}{q - q^{-1}} \end{aligned} \quad (68)$$

(we denoted  $d = x \partial_x$ ,  $x = z$  or  $w$ .) The Casimir operator for the quantum group:

$$q^{2J_q+1} + q^{-(2J_q+1)} = \frac{1}{2} [(q - q^{-1})^2 (ef + fe) + (q + q^{-1})(t + t^{-1})] \quad (69)$$

is an element of the center of  $\mathcal{U}_q(sl(2))$ . In the product of two representations, it acts as:

$$\begin{aligned} q^{2J_q+1} + q^{-(2J_q+1)} &= \\ &= q + q^{-1} + \left[ \left( \frac{qw}{z} + \frac{z}{qw} \right) - (q + q^{-1}) \right] (t_z - 1)(t_w^{-1} - 1) \end{aligned} \quad (70)$$

or, to the fourth order in  $\hbar$ :

$$2 + \hbar^2(1 + 4J(J+1)) + \hbar^3[-4(d_z - d_w)J(J+1) - 8\mathcal{T}^3] + \hbar^4\left[\frac{1}{12} - J(J+1)[4d_z d_w - \frac{8}{3}(d_z^2 + d_w^2) - 2] - 8\mathcal{T}^3(d_z - d_w)\right] + \dots \quad (71)$$

where we used the Yangian generator  $\mathcal{T}^3 = \frac{1}{2}(z^2 - w^2)\partial_z\partial_w$ .

The operator (70) has the same spectrum as  $q^{2J+1} + q^{-2J-1}$ , where  $J$  is the classical momentum. It is natural to conjecture that these operators are conjugate. Indeed, we found to the order  $\hbar^4$ :

$$q^{2J_q+1} + q^{-2J_q-1} = U(q^{2J+1} + q^{-2J-1})U^{-1} \quad (72)$$

where

$$U = \exp\left[\frac{\hbar}{2}(z^2 - w^2)\partial_z\partial_w - \frac{\hbar^2}{12}(z^2 - w^2)\partial_z\partial_w(d_z - d_w) + o(\hbar^2)\right] \quad (73)$$

Now

$$H = \frac{1}{2}[U(\psi_{q^2}(-J) + \psi_{q^2}(J+1))U^{-1} + (\hbar \rightarrow -\hbar)] \quad (74)$$

Notice that the coefficient of  $\hbar$  in  $\psi_{q^2}(-J) + \psi_{q^2}(J+1)$  equals to the commutator

$$\begin{aligned} & \left[\frac{1}{2}(z^2 - w^2)\partial_z\partial_w, H_0\right] = \\ & = [\mathcal{T}^3, H_0] = \left(w\frac{\partial}{\partial w} - z\frac{\partial}{\partial z}\right) \end{aligned} \quad (75)$$

in agreement with eq. (35). After adding the permuted (or  $\hbar \rightarrow -\hbar$ ) expression, this term cancels.

Let us find the coefficient of  $\hbar^2$  in  $H$ . Because of the property (72), it remains to calculate the expression  $\psi_{q^2}(-J) + \psi_{q^2}(J+1)$  where  $J$  is the classical  $J$ , to the order  $\hbar^2$ .

Let us collect here a few properties of the quantum  $\psi$ -function. Consider the function:

$$f_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-q^n} = \sum_{m=0}^{\infty} \frac{zq^m}{1-zq^m} \quad (76)$$

Write

$$\frac{1}{1-q^n} = -\frac{1}{\log q} \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \frac{(q^\epsilon; q)_n}{(q; q)_n} \quad (77)$$

where  $(a; q)_n = (1-a)(1-qa)\dots(1-q^{n-1}a)$ , and use the "quantum binomial formula"

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty} \quad (78)$$

and the definition of the  $\Gamma$ -function:

$$\Gamma_q(x) = \frac{(a; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x} \quad (79)$$

We have :

$$f_q(z) = \frac{\log(1-q)}{\log q} + \frac{1}{\log q} \psi_q(x) \quad (80)$$

Consider the function ([16], Ex. 1.21):

$$\begin{aligned}\frac{d}{dx}\psi_q(x) &= (\log q)^2 \sum_{n=0}^{\infty} \frac{q^{n+x}}{(1-q^{n+x})^2} = \\ &= \sum_{n=0}^{\infty} \frac{\hbar^2}{2 \cosh((n+x)\hbar)-2}\end{aligned}\quad (81)$$

Take the derivative of this function with respect to  $\hbar$ :

$$\begin{aligned}&\frac{\partial}{\partial \hbar} \sum_{n=0}^{\infty} \frac{\hbar^2}{2 \cosh((n+x)\hbar)-2} = \\ &= \sum_{n=0}^{\infty} \hbar \left( (n+x) \frac{\partial}{\partial(n+x)} + 2 \right) \frac{1}{2 \cosh \hbar(n+x)-2} = \\ &= \int_{\hbar x}^{\infty} dz (z \partial_z + 2) \frac{1}{2 \cosh z - 2} + \frac{\hbar}{2} \left( x \frac{d}{dx} + 2 \right) \frac{1}{2 \cosh \hbar x - 2}\end{aligned}\quad (82)$$

where we have used the formula

$$\hbar \sum_{n=0}^{\infty} f(x + n\hbar) = \int_x^{\infty} f(z) dz + \frac{\hbar}{2} f(x) + o(\hbar) \quad (83)$$

Taking the integral, we have:

$$\psi_{q^2}(-J) + \psi_{q^2}(J+1) = \psi(-J) + \psi(J+1) + \frac{\hbar^2}{3} J(J+1) \quad (84)$$

and

$$\begin{aligned}&\psi_{q^2}(-J_q) + \psi_{q^2}(J_q+1) = H_0 + \\ &\quad + \frac{\hbar}{2} [(z^2 - w^2) \partial_z \partial_w, H_0] + \\ &+ \frac{\hbar^2}{8} [(z^2 - w^2) \partial_z \partial_w, [(z^2 - w^2) \partial_z \partial_w, H_0]] - \\ &\quad - \frac{\hbar^2}{12} [(z^2 - w^2) \partial_z \partial_w (d_z - d_w), H_0] - \\ &\quad - \frac{\hbar^2}{3} (z - w)^2 \partial_z \partial_w\end{aligned}\quad (85)$$

where we denoted by  $H_0$  the XXX hamiltonian (66).

$$\begin{aligned}[(z - w) \partial_z \partial_w, H_0] &= \partial_w - \partial_z, \\ [(z^2 - w^2) \partial_z \partial_w, H_0] &= 2(d_w - d_z)\end{aligned}\quad (86)$$

The terms of the second order in  $\hbar$ :

We get the term

$$\frac{\hbar^2}{2} (z^2 + w^2) \partial_z \partial_w$$

from the double commutator.

We also get the term  $\frac{\hbar^2}{6} (d_z - d_w)^2$ .

The most complicated expression arises from

$$(z^2 - w^2) \partial_z \partial_w [(d_z - d_w), H_0] \quad (87)$$

We have:

$$H_0 = \log(\partial_z \partial_w) + \frac{1}{\partial_z} \log(z - w) \partial_z + \frac{1}{\partial_w} \log(z - w) \partial_w \quad (88)$$

and

$$\begin{aligned}&(z^2 - w^2) \partial_z \partial_w [(d_z - d_w), H_0] = \\ &= (z^2 - w^2) \partial_z \partial_w \left[ \frac{1}{\partial_z} \frac{z+w}{z-w} \partial_z + \frac{1}{\partial_w} \frac{z+w}{z-w} \partial_w \right] = \\ &= (z^2 - w^2) \left[ \partial_w \frac{z+w}{z-w} \partial_z + \partial_z \frac{z+w}{z-w} \partial_w \right]\end{aligned}\quad (89)$$

So we have:

$$H = H_0 + \hbar^2 \left[ \frac{1}{2}(z^2 + w^2)\partial_z\partial_w + \frac{1}{6}(d_z - d_w)^2 - \frac{1}{12}(z^2 - w^2) \left( \partial_w \frac{z+w}{z-w} \partial_z + \partial_z \frac{z+w}{z-w} \partial_w \right) - \frac{1}{3}(z - w)^2 \partial_z \partial_w \right] \quad (90)$$

An interesting property of this equation is its locality in the coordinate space: it does not contain operators like  $1/\partial$ . The most singular (when  $z \rightarrow w$ ) term reminds 2D fermionic propagator. Notice that this expression is not translational invariant. Indeed, the translational invariance is the commutativity with the operator  $s^+ = \partial_z + \partial_w$ . In the  $XXZ$  case, this operator is deformed (see Appendix), and thus  $XXZ$  hamiltonian is not translational-invariant<sup>6</sup>

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## Appendix.

The  $\mathcal{U}_q(\widehat{sl}(2))$ -algebra is defined by the generators

$$e_0, e_1, f_0, f_1, t_0, t_1 \quad (91)$$

with the commutation relations:

$$\begin{aligned} t_i e_j t_i^{-1} &= q^{a_{ij}} e_j \\ t_i f_j t_i^{-1} &= q^{-a_{ij}} f_j \\ [e_i, f_j] &= \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}} \end{aligned} \quad (92)$$

where

$$a_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad (93)$$

and the Serre relations:

$$x_i^3 x_j - [3] x_i^2 x_j x_i + [3] x_i x_j x_i^2 - x_j x_i^3 = 0 \quad (94)$$

where  $i \neq j$  and  $x = e, f$ .

The "quantum adjoint action":

$$\text{ad}_e(A) = \sum_i e^i A s(e_i) \quad (95)$$

where

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<sup>6</sup>To avoid confusion, we remind that  $H_{XXZ}$  does not commute with  $U_q(sl(2))$ . Indeed, it was obtained from the  $U_q(sl(2))$ -invariant expression by symmetrization.

$$\sum_i e^i \otimes e_i = \Delta(e) \quad (96)$$

The comultiplication in  $\mathcal{U}_q(\widehat{sl}(2))$  is <sup>7</sup> :

$$\begin{aligned} \Delta(t_i) &= t_i \otimes t_i \\ \Delta(x_i) &= x_i \otimes t_i^{-1/2} + t_i^{1/2} \otimes x_i \end{aligned} \quad (97)$$

The other comultiplication,  $\Delta'$ , differs from this one by permuting tensor multipliers:  $\Delta' = P\Delta P$ . This is also homomorphism from the universal enveloping to its tensor square, since this fact obviously does not depend on the order of tensor multipliers.

The antipode is:

$$\begin{aligned} a(t_i) &= t_i^{-1} \\ a(e_i) &= -q^{-1}e_i \\ a(f_i) &= -qf_i \end{aligned} \quad (98)$$

where again  $x_i = e_i, f_i$ .

We can rewrite the Serre relations as:

$$\text{ad}_{x_i}^{1-a_{ij}}(x_j) = 0 \quad (99)$$

and  $e_0$  is the vector operator with respect to the  $\mathcal{U}_q(sl(2))$  subalgebra  $(e_1, f_1, t_1)$ .

We will need the evaluation representation of this algebra, which is defined as follows. Let  $\rho$  be some (finite-dimensional) representation of  $\mathcal{U}_q(sl(2))$ . Then the evaluation representation  $\rho_\zeta$  of  $\mathcal{U}_q(\widehat{sl}(2))$  depends on some parameter  $\zeta$ , and the space of this representation coincides with the space of  $\rho$  :

$$\begin{aligned} \rho_\zeta(x_1) &= \rho(x_1), \quad x = e, f \\ \rho_\zeta(e_0) &= \zeta \rho(f_1) \\ \rho_\zeta(f_0) &= \frac{1}{\zeta} \rho(e_1) \\ \rho_\zeta(t_0) &= (\rho_\zeta(t_1))^{-1} = (\rho(t_1))^{-1} \end{aligned} \quad (100)$$

The interesting question is what is the tensor product of two such evaluation representations. In general it is some irreducible representation, depending on two parameters. But for the special values of  $\zeta$ 's this product turns out to be reducible.

Example. Consider the product of two spin-1/2 representations, with the parameters  $z$  and  $w$ . Consider the vector  $\downarrow \otimes \downarrow$ . Acting on this vector by  $e_1$ , we get:

$$\begin{aligned} (q^{-\frac{\sigma^3}{2}} \otimes \sigma^+ + \sigma^+ \otimes q^{\frac{\sigma^3}{2}}) \downarrow \otimes \downarrow &= \\ = q^{1/2} \downarrow \otimes \uparrow + q^{-1/2} \uparrow \otimes \downarrow \end{aligned} \quad (101)$$

And acting on it by  $f_0$ , we get:

$$\begin{aligned} (q^{\frac{\sigma^3}{2}} \otimes \sigma^+ / z + \sigma^+ / w \otimes q^{-\frac{\sigma^3}{2}}) \downarrow \otimes \downarrow &= \\ = (q^{-1/2} / z) \downarrow \otimes \uparrow + (q^{1/2} / w) \uparrow \otimes \downarrow \end{aligned} \quad (102)$$

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<sup>7</sup>Our definition of  $e, f, t$  differs from the one in [15] by  $e_i = e^{[JM]_i} t^{[JM]_i^{-1/2}}, f_i = t^{[JM]_i^{1/2}} f^{[JM]_i}, t_i = t_i^{(JM)}$



These two vectors are different at general values of  $z$  and  $w$ , but when  $w = q^2 z$  (102) equals (101) multiplied by  $\frac{1}{qz}$ .

This means, that at this special value there is a 3-dimensional subrepresentation in this 4-dimensional tensor product.

In general,

$$\rho_{\zeta}^{1/2} \otimes \rho_{q^2 \zeta}^{1/2} \otimes \cdots \otimes \rho_{q^{2(n-1)} \zeta}^{1/2} \supset \rho_{q^{n-1} \zeta}^{n/2} \quad (103)$$

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